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Vanishing Theorems on Toric Varieties in Positive Characteristic *

Qihong Xie

Dedicated to Professor Yujiro Kawamata for his sixtieth birthday

Abstract

We use the liftability of the relative Frobenius morphism of toric varieties and the strong liftability of toric varieties to prove the Bott vanishing theorem, the degeneration of the Hodge to de Rham spectral sequence and the Kawamata-Viehweg vanishing theorem for log pairs on toric varieties in positive characteristic. These results generalize those results of Danilov, Buch-Thomsen-Lauritzen-Mehta, Mustaţă and Fujino to the case where concerned Weil divisors are not necessarily torus invariant.

1 Introduction

Throughout this paper, we always work over *a perfect field k of characteristic $p > 0$* unless otherwise stated. The main purpose of this paper is to develop various vanishing theorems on toric varieties in positive characteristic by means of the lifting technique, which consists of two points: one is the liftability of the relative Frobenius morphism of toric varieties, and the other is the strong liftability of toric varieties.

The following are the main theorems in this paper, which generalize those results of Danilov [Da78], Buch-Thomsen-Lauritzen-Mehta [BTLM97], Mustaţă [Mu02] and Fujino [Fu07] to the case where concerned Weil divisors are not necessarily torus invariant. See Definition 2.12 for the definition of $\widetilde{\Omega}^\bullet_X(\log D)$, the Zariski-de Rham complex of X with logarithmic poles along D.

Theorem 1.1 (Bott vanishing)**.** *Let X be a projective toric variety over k, D a reduced Weil divisor on X, and $\mathcal{L}$ an ample invertible sheaf on X. Then $H^j(X, \widetilde{\Omega}^i_X(\log D) \otimes \mathcal{L}) = 0$ holds for any $j > 0$ and any $i \geq 0$.*

Theorem 1.2 (Hodge to de Rham spectral sequence)**.** *Let X be a projective toric variety over k, and D a reduced Weil divisor on X. Then the Hodge to de Rham spectral sequence degenerates in E_1:*

$$E_1^{ij} = H^j(X, \widetilde{\Omega}^i_X(\log D)) \Longrightarrow \mathbf{H}^{i+j}(X, \widetilde{\Omega}^\bullet_X(\log D)).$$

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Theorem 1.3 (Kawamata-Viehweg vanishing)**.** *Let X be a projective simplicial toric variety over k, and H an ample $\mathbb{Q}$-divisor on X. Then $H^i(X, K_X + \ulcorner H \urcorner) = 0$ holds for any $i > 0$.*

All of the results are fresh and should be of the strongest form for toric varieties in positive characteristic. Toric varieties are elementary, however, we can construct some interesting classes of algebraic varieties in positive characteristic from toric varieties. It turns out that the celebrated Kawamata-Viehweg vanishing theorem [Ka82, Vi82] plays an essential role in birational geometry of complex algebraic varieties. Hence Theorem 1.3 is also expected to be helpful in the study of algebraic varieties in positive characteristic.

Furthermore, we prove more general results as Theorems 3.1, 3.2, and 3.6 for a normal projective variety whose relative Frobenius morphism has a global lifting and which is strongly liftable over $W_2(k)$. We also give an example as Corollary 3.9 to show that the main theorems could hold for more general varieties.

In §2, we will recall some definitions and preliminary results. §3 is devoted to the proofs of the main theorems. For the necessary notions and results in toric geometry, we refer the reader to [Da78], [Od88], [Fu93] and [CLH11].

Notation. We use $[B] = \sum [b_i] B_i$ (resp. $\ulcorner B \urcorner = \sum \ulcorner b_i \urcorner B_i$, $\langle B \rangle = \sum \langle b_i \rangle B_i$) to denote the round-down (resp. round-up, fractional part) of a $\mathbb{Q}$-divisor $B = \sum b_i B_i$, where for a real number b, $[b] := \max\{n \in \mathbb{Z} \,|\, n \leq b\}$, $\ulcorner b \urcorner := -[-b]$ and $\langle b \rangle := b - [b]$.

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2 Preliminaries

Definition 2.1. The ring of Witt vectors of length two of k, denoted by $W_2(k)$, is $k \oplus k$ as set, where addition and multiplication for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ in $W_2(k)$ are defined by

$$\begin{aligned} a + b &= (a_0 + b_0, a_1 + b_1 - \frac{1}{p} \sum_{0<i<p} \binom{p}{i} a_0^i b_0^{p-i}), \\ a \cdot b &= (a_0 b_0, a_0^p b_1 + b_0^p a_1). \end{aligned}$$

We can prove that $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$, $W_2(k) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = k$ and $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$ if $k = \mathbb{Z}/p\mathbb{Z}$. In fact, the projection $pr_1 : W_2(k) \to k$ given by $pr_1(a_0, a_1) = a_0$ corresponds to the reduction $W_2(k)/pW_2(k) = k$ modulo p, and the ring homomorphism $F_{W_2(k)} : W_2(k) \to W_2(k)$ given by $F_{W_2(k)}(a_0, a_1) = (a_0^p, a_1^p)$ reduces to the Frobenius homomorphism $F_k : k \to k$ modulo p. We refer the reader to [Se62, II.6] for more details.

The following definition [EV92, Definition 8.11] generalizes the definition [DI87, 1.6] of liftings of k-schemes over $W_2(k)$.

Definition 2.2. Let X be a noetherian scheme over k, and $D = \sum_i D_i$ a reduced Cartier divisor on X. A lifting of (X, D) over $W_2(k)$ consists of a scheme $\widetilde{X}$ and closed subschemes $\widetilde{D}_i \subset \widetilde{X}$, all defined and flat over $W_2(k)$ such that $X = \widetilde{X} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$ and $D_i = \widetilde{D}_i \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$. We write $\widetilde{D} = \sum_i \widetilde{D}_i$ and say that $(\widetilde{X}, \widetilde{D})$ is a lifting of (X, D) over $W_2(k)$, if no confusion is likely.

Let $\widetilde{X}$ be a lifting of X over $W_2(k)$. Then we have an exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules:

$$0 \to p \cdot \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}} \xrightarrow{r} \mathcal{O}_X \to 0, \tag{1}$$

together with an $\mathcal{O}_{\widetilde{X}}$-module isomorphism

$$p : \mathcal{O}_X \to p \cdot \mathcal{O}_{\widetilde{X}}, \tag{2}$$

where r is the reduction modulo p satisfying $p(x) = p\widetilde{x}$, $r(\widetilde{x}) = x$ for $x \in \mathcal{O}_X$, $\widetilde{x} \in \mathcal{O}_{\widetilde{X}}$.

Assume that X is smooth over k and $D = \sum_i D_i$ is simple normal crossing. If $(\widetilde{X}, \widetilde{D})$ is a lifting of (X, D) over $W_2(k)$, then $\widetilde{X}$ is smooth over $W_2(k)$ and $\widetilde{D} = \sum_i \widetilde{D}_i$ is relatively simple normal crossing over $W_2(k)$, i.e. $\widetilde{X}$ is covered by affine open subsets $\{U_\alpha\}$, such that each U_α is étale over $\mathbb{A}^n_{W_2(k)}$ via coordinates $\{x_1, \cdots, x_n\}$ and $\widetilde{D}|_{U_\alpha}$ is defined by the equation $x_1 \cdots x_\nu = 0$ with $1 \leq \nu \leq n$ (see [EV92, Lemmas 8.13, 8.14]). Therefore we have an exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules:

$$0 \to p \cdot \Omega^1_{\widetilde{X}/W_2(k)}(\log \widetilde{D}) \to \Omega^1_{\widetilde{X}/W_2(k)}(\log \widetilde{D}) \xrightarrow{r} \Omega^1_X(\log D) \to 0,$$

together with an $\mathcal{O}_{\widetilde{X}}$-module isomorphism $p : \Omega^1_X(\log D) \to p \cdot \Omega^1_{\widetilde{X}/W_2(k)}(\log \widetilde{D})$.

Definition 2.3. Let X be a noetherian scheme over k, $F_X : X \to X$ the absolute Frobenius morphism of X, and $X' = X \times_{\operatorname{Spec} k} F_k$ the fiber product. Then we have the following commutative diagram with cartesian square:

$$\begin{tikzcd}
X \arrow[r, "F"'] \arrow[rr, bend left, "F_X"] \arrow[dr] & X' \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "F_k"] & \operatorname{Spec} k.
\end{tikzcd}$$

The induced k-morphism $F : X \to X'$ is called the relative Frobenius morphism of X/k. Assume that X has a lifting $\widetilde{X}$ over $W_2(k)$. Define $\widetilde{X}' = \widetilde{X} \times_{\operatorname{Spec} W_2(k)} F_{W_2(k)}$. A $W_2(k)$-morphism $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ is called a lifting of the relative Frobenius morphism of X/k, if the restriction of $\widetilde{F}$ to X is just $F : X \to X'$.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{F}"] \arrow[dr] & \widetilde{X}' \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
& \operatorname{Spec} W_2(k) \arrow[r, "F_{W_2(k)}"] & \operatorname{Spec} W_2(k).
\end{tikzcd}$$

Let D be a reduced Cartier divisor on X, $\widetilde{D}$ a lifting of D over $W_2(k)$ and $\widetilde{D}' = \widetilde{D} \times_{\operatorname{Spec} W_2(k)} F_{W_2(k)}$ the divisor on $\widetilde{X}'$. $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ is said to be compatible with $\widetilde{D}$, if $\widetilde{F}^* \mathcal{O}_{\widetilde{X}'}(-\widetilde{D}') = \mathcal{O}_{\widetilde{X}}(-p\widetilde{D})$ holds.

The following theorem is due to Deligne and Illusie, although its explicit statement and proof have not been given in [DI87].

Theorem 2.4. *Let X be a smooth scheme over k, and $D = \sum_i D_i$ a simple normal crossing divisor on X. Assume that (X, D) has a lifting $(\widetilde{X}, \widetilde{D})$ over $W_2(k)$ and the relative Frobenius morphism $F : X \to X'$ has a lifting $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$, which is compatible with $\widetilde{D}$. Then there is a quasi-isomorphism of complexes of $\mathcal{O}_{X'}$-modules:*

$$\phi : \bigoplus_i \Omega^i_{X'}(\log D')[-i] \to F_*\Omega^\bullet_X(\log D).$$

Proof. We use a similar proof to that of [DI87, Théorème 2.1]. Since $F^* : \Omega^1_{X'}(\log D') \to F_*\Omega^1_X(\log D)$ is trivial, the image of $\widetilde{F}^* : \Omega^1_{\widetilde{X}'/W_2(k)}(\log \widetilde{D}') \to \widetilde{F}_*\Omega^1_{\widetilde{X}/W_2(k)}(\log \widetilde{D})$ is contained in $p \cdot \widetilde{F}_*\Omega^1_{\widetilde{X}/W_2(k)}(\log \widetilde{D})$. Therefore, there exists a unique homomorphism $f = p^{-1}\widetilde{F}^* : \Omega^1_{X'}(\log D') \to F_*\Omega^1_X(\log D)$ making the following diagram commutative:

$$\begin{tikzcd}
\Omega^1_{\widetilde{X}'/W_2(k)}(\log \widetilde{D}') \arrow[r, "\widetilde{F}^*"] \arrow[d, "r"'] & p \cdot \widetilde{F}_*\Omega^1_{\widetilde{X}/W_2(k)}(\log \widetilde{D}) \\
\Omega^1_{X'}(\log D') \arrow[r, "f"] & F_*\Omega^1_X(\log D). \arrow[u, "p", "\cong"']
\end{tikzcd}$$

Let $\widetilde{x}$ be a local section of $\mathcal{O}_{\widetilde{X}}(-\widetilde{D})$ and $x = r(\widetilde{x})$ the induced local section of $\mathcal{O}_X(-D)$. Since $\widetilde{F}$ is compatible with $\widetilde{D}$, we have $\widetilde{F}^*(\widetilde{x} \otimes 1) = \widetilde{x}^p(1 + pu(\widetilde{x}))$, where $u(\widetilde{x})$ is a local section of $\mathcal{O}_{\widetilde{X}}$. Hence we have

$$f(\frac{dx \otimes 1}{x \otimes 1}) = \frac{dx}{x} + du(\widetilde{x}). \tag{3}$$

In particular, we have $df = 0$, hence f induces a homomorphism $f : \Omega^1_{X'}(\log D') \to \mathcal{Z}^1(F_*\Omega^\bullet_X(\log D))$. Define $\phi^0 : \mathcal{O}_{X'} \to F_*\mathcal{O}_X$ to be the Frobenius homomorphism. For any $i \geq 1$, define $\phi^i = \wedge^i f : \Omega^i_{X'}(\log D') \to \mathcal{Z}^i(F_*\Omega^\bullet_X(\log D))$ to be the i-th wedge product of f. By abuse of notation, we denote $\phi^i : \Omega^i_{X'}(\log D') \to F_*\Omega^i_X(\log D)$ to be the composition of $\wedge^i f$ with the natural inclusion $\mathcal{Z}^i(F_*\Omega^\bullet_X(\log D)) \hookrightarrow F_*\Omega^i_X(\log D)$. Thus we have a morphism of complexes of $\mathcal{O}_{X'}$-modules:

$$\phi = \bigoplus_i \phi^i : \bigoplus_i \Omega^i_{X'}(\log D')[-i] \to F_*\Omega^\bullet_X(\log D).$$

It follows from (3) and the definition of ϕ^i that $\mathcal{H}^i(\phi) = C^{-1}$ holds for any $i \geq 0$, where $C : \mathcal{H}^i(F_*\Omega^\bullet_X(\log D)) \to \Omega^i_{X'}(\log D')$ is the Cartier isomorphism (cf. [Ka70, 7.2]). Thus ϕ is a quasi-isomorphism of complexes of $\mathcal{O}_{X'}$-modules. □

We can generalize [BTLM97, Theorem 2] to the logarithmic case.

Proposition 2.5. *Notation and assumptions as in Theorem 2.4, then the homomorphism $\phi^i : \Omega^i_{X'}(\log D') \to F_*\Omega^i_X(\log D)$ is a split injection for any $i \geq 0$.*

Proof. Denote $n = \dim X$, $\mathcal{B}^i = \mathcal{B}^i(F_*\Omega^\bullet_X(\log D))$, $\mathcal{Z}^i = \mathcal{Z}^i(F_*\Omega^\bullet_X(\log D))$. By using the Cartier isomorphism [Ka70, 7.2], we have the following exact sequence for any $i \geq 0$:

$$0 \to \mathcal{B}^i \to \mathcal{Z}^i \xrightarrow{C} \Omega^i_{X'}(\log D') \to 0. \tag{4}$$

In fact, the exact sequence (4) splits since $C\phi^i = \mathrm{Id}$ holds for any $i \geq 0$.

Define a homomorphism $\psi^i : F_*\Omega^i_X(\log D) \to \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^{n-i}_{X'}(\log D'), \Omega^n_{X'}(\log D'))$ by $\omega \mapsto \psi^i(\omega)$, where $\psi^i(\omega)(\eta) = C(\phi^{n-i}(\eta) \wedge \omega)$ and $C : F_*\Omega^n_X(\log D) = \mathcal{Z}^n \to \Omega^n_{X'}(\log D')$ is the homomorphism in the exact sequence (4) for $i = n$. Let z be a local section of $\Omega^i_{X'}(\log D')$. Then we have $\psi^i(\phi^i(z))(\eta) = C(\phi^{n-i}(\eta) \wedge \phi^i(z)) = C(\phi^n(\eta \wedge z)) = \eta \wedge z$. By using the perfect pairing between $\Omega^{n-i}_{X'}(\log D')$ and $\Omega^i_{X'}(\log D')$ given by the wedge product, we obtain a well-defined homomorphism $\psi^i : F_*\Omega^i_X(\log D) \to \Omega^i_{X'}(\log D')$ satisfying $\psi^i\phi^i = \mathrm{Id}$, which is the desired splitting of ϕ^i. $\square$

In order to deal with $\mathbb{Q}$-divisors, we need the following lemma due to Hara [Ha98].

Lemma 2.6. *Let X be a smooth scheme over k, $D = \sum_i D_i$ a simple normal crossing divisor on X, and $G = \sum_i r_i D_i$ an effective divisor on X with $0 \leq r_i < p$ for any i. Then the inclusion of complexes $\Omega^\bullet_X(\log D) \hookrightarrow \Omega^\bullet_X(\log D)(G) := \Omega^\bullet_X(\log D) \otimes \mathcal{O}_X(G)$ induces a quasi-isomorphism of complexes of $\mathcal{O}_{X'}$-modules:*

$$\nu : F_*\Omega^\bullet_X(\log D) \hookrightarrow F_*(\Omega^\bullet_X(\log D)(G)).$$

Proof. By Künneth's formula, we can reduce to prove the case where $\dim X = 1$, D is defined by the local parameter t and $G = rD$ with $0 \leq r < p$. In this case, the complex of $\mathcal{O}_{X'}$-modules $F_*(\Omega^\bullet_X(\log D)(G))$ is written as:

$$0 \to \bigoplus_{i=0}^{p-1} \mathcal{O}_{X'} \cdot t^i \cdot \frac{1}{t^r} \xrightarrow{d} \bigoplus_{i=0}^{p-1} \mathcal{O}_{X'} \cdot \frac{dt}{t} \cdot t^i \cdot \frac{1}{t^r} \to 0.$$

By an easy calculation, we have $\mathcal{H}^0 = \mathcal{O}_{X'}$ and $\mathcal{H}^1 = \mathcal{O}_{X'} \cdot dt/t$, which is independent of r. Hence ν is a quasi-isomorphism. $\square$

With the same notation and assumptions as in Theorem 2.4 and Lemma 2.6, define ϕ^i_G to be the composition of the following homomorphisms for any $i \geq 0$:

$$\phi^i_G : \Omega^i_{X'}(\log D') \xrightarrow{\phi^i} \mathcal{Z}^i(F_*\Omega^\bullet_X(\log D)) \xrightarrow{\nu} \mathcal{Z}^i(F_*(\Omega^\bullet_X(\log D)(G))) \hookrightarrow F_*(\Omega^i_X(\log D)(G)).$$

Since $\phi^i \wedge \phi^j = \phi^{i+j}$ holds for any $i, j \geq 0$, by using the natural homomorphism $\wedge : \Omega^i_X(\log D) \otimes \Omega^j_X(\log D)(G) \to \Omega^{i+j}_X(\log D)(G)$, we can show that $\phi^i \wedge \phi^j_G = \phi^{i+j}_G$ holds for any $i, j \geq 0$. Denote the composition of $\mathcal{H}^i(\nu)^{-1} : \mathcal{H}^i(F_*\Omega^\bullet_X(\log D)(G)) \to \mathcal{H}^i(F_*\Omega^\bullet_X(\log D))$ with the Cartier isomorphism $C : \mathcal{H}^i(F_*\Omega^\bullet_X(\log D)) \to \Omega^i_{X'}(\log D')$ by $C_G : \mathcal{H}^i(F_*(\Omega^\bullet_X(\log D)(G))) \to \Omega^i_{X'}(\log D')$. Then the morphism

$$\phi_G = \bigoplus_i \phi^i_G : \bigoplus_i \Omega^i_{X'}(\log D')[-i] \to F_*(\Omega^\bullet_X(\log D)(G))$$

is a quasi-isomorphism since it induces the isomorphism $C_G^{-1} = \mathcal{H}^i(\nu)C^{-1}$ on each $\mathcal{H}^i$.

Proposition 2.7. *Let X be a smooth scheme over k, $D = \sum_i D_i$ a simple normal crossing divisor on X, and $G = \sum_i r_i D_i$ an effective divisor on X with $0 \leq r_i < p$ for any i. Assume that (X, D) has a lifting $(\widetilde{X}, \widetilde{D})$ over $W_2(k)$ and the relative Frobenius morphism $F : X \to X'$ has a lifting $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$, which is compatible with $\widetilde{D}$. Then $\phi^i_G : \Omega^i_{X'}(\log D') \to F_*(\Omega^i_X(\log D)(G))$ is a split injection for any $i \geq 0$.*

Proof. Denote $n = \dim X$, $\mathcal{B}^i_G = \mathcal{B}^i(F_*(\Omega^\bullet_X(\log D)(G)))$, $\mathcal{Z}^i_G = \mathcal{Z}^i(F_*(\Omega^\bullet_X(\log D)(G)))$. By using the isomorphism C_G, we have the following exact sequence for any $i \geq 0$:

$$0 \to \mathcal{B}^i_G \to \mathcal{Z}^i_G \stackrel{C_G}{\to} \Omega^i_{X'}(\log D') \to 0. \tag{5}$$

In fact, the exact sequence (5) splits since $C_G\phi^i_G = \mathrm{Id}$ holds for any $i \geq 0$.

Define a homomorphism

$$\psi^i_G : F_*(\Omega^i_X(\log D)(G)) \to \mathcal{H}om_{\mathcal{O}_{X'}}(\Omega^{n-i}_{X'}(\log D'), \Omega^n_{X'}(\log D'))$$

by $\omega \mapsto \psi^i_G(\omega)$, where $\psi^i_G(\omega)(\eta) = C_G(\phi^{n-i}(\eta) \wedge \omega)$ and $C_G : F_*(\Omega^n_X(\log D)(G)) = \mathcal{Z}^n_G \to \Omega^n_{X'}(\log D')$ is the homomorphism in the exact sequence (5) for $i = n$. Let z be a local section of $\Omega^i_{X'}(\log D')$. Then we have $\psi^i_G(\phi^i_G(z))(\eta) = C_G(\phi^{n-i}(\eta) \wedge \phi^i_G(z)) = C_G(\phi^n_G(\eta \wedge z)) = \eta \wedge z$. By using the perfect pairing between $\Omega^{n-i}_{X'}(\log D')$ and $\Omega^i_{X'}(\log D')$ given by the wedge product, we obtain a well-defined homomorphism $\psi^i_G : F_*(\Omega^i_X(\log D)(G)) \to \Omega^i_{X'}(\log D')$ satisfying $\psi^i_G\phi^i_G = \mathrm{Id}$, which is the desired splitting of ϕ^i_G. $\square$

Theorem 2.8. *Notation and assumptions as in Proposition 2.7, let H be a $\mathbb{Q}$-divisor on X with $\mathrm{Supp}(\langle H \rangle) \subseteq D$. Then there exists a quasi-isomorphism of complexes of $\mathcal{O}_{X'}$-modules:*

$$\phi_H = \bigoplus_i \phi^i_H : \bigoplus_i \Omega^i_{X'}(\log D')(-\ulcorner H' \urcorner)[-i] \to F_*(\Omega^\bullet_X(\log D)(-\ulcorner pH \urcorner)).$$

In fact, $\phi^i_H : \Omega^i_{X'}(\log D')(-\ulcorner H' \urcorner) \to F_*(\Omega^i_X(\log D)(-\ulcorner pH \urcorner))$ is a split injection for any $i \geq 0$.

Proof. Let $G = p\ulcorner H \urcorner - \ulcorner pH \urcorner$. Then it is easy to see that G satisfies the condition in Proposition 2.7. Tensoring ϕ^i_G by $\mathcal{O}_{X'}(-\ulcorner H' \urcorner)$, we can obtain the conclusions. $\square$

There is a slight generalization of [Xie10, Definition 2.3].

Definition 2.9. Let X be a noetherian scheme over k. X is said to be strongly liftable over $W_2(k)$, if the following two conditions hold:

(i) X has a lifting $\widetilde{X}$ over $W_2(k)$;

(ii) For any effective Cartier divisor D on X, there is a lifting $\widetilde{D}$ of D over $W_2(k)$ in the following sense: D is regarded as a closed subscheme of X, and $\widetilde{D}$ is a closed subscheme of the fixed scheme $\widetilde{X}$ such that $\widetilde{D}$ is flat over $W_2(k)$ and $\widetilde{D} \times_{\mathrm{Spec}\, W_2(k)} \mathrm{Spec}\, k = D$.

If X is a smooth scheme over k, then Definition 2.9 is equivalent to [Xie10, Definition 2.3]. Furthermore, it was proved in [Xie10] that $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, smooth projective curves, smooth rational surfaces, and certain smooth complete intersections in $\mathbb{P}^n_k$ are strongly liftable over $W_2(k)$.

Let X be a noetherian scheme over k, $\widetilde{X}$ a lifting of X over $W_2(k)$, $\mathcal{L}$ an invertible sheaf on X and $\widetilde{\mathcal{L}}$ a lifting of $\mathcal{L}$ on $\widetilde{X}$, i.e. $\widetilde{\mathcal{L}}$ is an invertible sheaf on $\widetilde{X}$ such that $\widetilde{\mathcal{L}}|_X = \mathcal{L}$. Tensoring (1) by $\widetilde{\mathcal{L}}$ and taking cohomology groups, we obtain the following exact sequence:

$$0 \to H^0(\widetilde{X}, p \cdot \widetilde{\mathcal{L}}) \to H^0(\widetilde{X}, \widetilde{\mathcal{L}}) \stackrel{r}{\to} H^0(X, \mathcal{L}) \to H^1(\widetilde{X}, p \cdot \widetilde{\mathcal{L}}). \tag{6}$$

We recall here a sufficient condition for strong liftability of schemes (cf. [Xie10, Proposition 3.4]).

Proposition 2.10. *Let X be a noetherian scheme over k, and $\widetilde{X}$ a lifting of X over $W_2(k)$. Assume that for any effective Cartier divisor D on X, the associated invertible sheaf $\mathcal{L} = \mathcal{O}_X(D)$ has a lifting $\widetilde{\mathcal{L}}$ on $\widetilde{X}$, and the natural map $r : H^0(\widetilde{X}, \widetilde{\mathcal{L}}) \to H^0(X, \mathcal{L})$ is surjective. Then X is strongly liftable over $W_2(k)$.*

Proof. Let D be an effective Cartier divisor on X, $\mathcal{L} = \mathcal{O}_X(D)$ the associated invertible sheaf on X, and $s \in H^0(X, \mathcal{L})$ the corresponding nonzero section. By assumption, $\mathcal{L}$ has a lifting $\widetilde{\mathcal{L}}$ on $\widetilde{X}$, and the natural map $r : H^0(\widetilde{X}, \widetilde{\mathcal{L}}) \to H^0(X, \mathcal{L})$ is surjective, hence there is a nonzero section $\widetilde{s} \in H^0(\widetilde{X}, \widetilde{\mathcal{L}})$ with $r(\widetilde{s}) = s$. Define $\widetilde{D} = \mathrm{div}_0(\widetilde{s})$ to be the associated effective Cartier divisor on $\widetilde{X}$. Then it is easy to see that $\widetilde{D}$ is a lifting of D over $W_2(k)$. □

Let X be a normal scheme over k, and D a reduced Weil divisor on X. Then there exists an open subset U of X such that $\mathrm{codim}_X(X - U) \geq 2$, U is smooth over k, and $D|_U$ is simple normal crossing on U. Such an U is called a required open subset for the log pair (X, D).

Example 2.11. (i) Let X be a smooth scheme over k, and D a simple normal crossing divisor on X. Then we can take the largest required open subset $U = X$.

(ii) Let $X = X(\Delta, k)$ be a toric variety, and D a torus invariant reduced Weil divisor on X. Take U to be $X(\Delta_1, k)$, where Δ_1 is the fan consisting of all 1-dimensional cones in Δ. Then it is easy to show that U is an open subset of X with $\mathrm{codim}_X(X - U) \geq 2$, U is smooth over k, and $D|_U$ is simple normal crossing on U. Hence U is a required open subset for (X, D).

Definition 2.12. Let X be a normal scheme over k, and D a reduced Weil divisor on X. Take a required open subset U for (X, D), and denote $\iota : U \hookrightarrow X$ to be the open immersion. For any $i \geq 0$, define the Zariski sheaf of differential i-forms of X with logarithmic poles along D by

$$\widetilde{\Omega}^i_X(\log D) = \iota_* \Omega^i_U(\log D|_U).$$

Since $\mathrm{codim}_X(X - U) \geq 2$ and $\Omega^i_U(\log D|_U)$ is a locally free sheaf on U, $\widetilde{\Omega}^i_X(\log D)$ is a reflexive sheaf on X, and the definition of $\widetilde{\Omega}^i_X(\log D)$ is independent of the choice of the open subset U. Furthermore we have the Zariski-de Rham complex $(\widetilde{\Omega}^\bullet_X(\log D), d)$.

3 Proofs of the main theorems

First of all, we will prove some more general results, which imply the main theorems.

Theorem 3.1. *Let X be a normal projective variety over k, D a reduced Weil divisor on X, and U a required open subset for (X, D). Assume that $(U, D|_U)$ has a lifting $(\widetilde{U}, \widetilde{D|_U})$ over $W_2(k)$ and $F : U \to U'$ has a lifting $\widetilde{F} : \widetilde{U} \to \widetilde{U}'$, which is compatible with $\widetilde{D|_U}$. Then $H^j(X, \widetilde{\Omega}^i_X(\log D) \otimes \mathcal{L}) = 0$ holds for any $j > 0$, any $i \geq 0$ and any ample invertible sheaf $\mathcal{L}$ on X.*

Proof. By assumption and Proposition 2.5, we have a split injection $\phi^i : \Omega^i_{U'}(\log D'|_{U'}) \to F_*\Omega^i_U(\log D|_U)$ for any $i \geq 0$. Let $\iota : U \hookrightarrow X$ and $\iota' : U' \hookrightarrow X'$ be the open immersions. Applying ι'_* to ϕ^i, since $\iota'_* F_* = F_* \iota_*$, we have a split injection for any $i \geq 0$:

$$\iota'_*\phi^i : \widetilde{\Omega}^i_{X'}(\log D') \to F_*\widetilde{\Omega}^i_X(\log D).$$

Tensoring $\iota'_*\phi^i$ by $\mathcal{L}'$ and using the projection formula, since k is perfect, we obtain an injection for any $j > 0$:

$$H^j(X, \widetilde{\Omega}^i_X(\log D) \otimes \mathcal{L}) = H^j(X', \widetilde{\Omega}^i_{X'}(\log D') \otimes \mathcal{L}') \hookrightarrow H^j(X, \widetilde{\Omega}^i_X(\log D) \otimes \mathcal{L}^p).$$

Iterating these injections, we obtain the desired vanishing by Serre's vanishing. $\square$

Associated with the Zariski-de Rham complex $\widetilde{\Omega}^\bullet_X(\log D)$, there is a spectral sequence:

$$E_1^{ij} = H^j(X, \widetilde{\Omega}^i_X(\log D)) \Longrightarrow \mathbf{H}^{i+j}(X, \widetilde{\Omega}^\bullet_X(\log D)), \tag{7}$$

where $\mathbf{H}^\bullet(X, \widetilde{\Omega}^\bullet_X(\log D))$ denotes the hypercohomology of the complex $\widetilde{\Omega}^\bullet_X(\log D)$. (7) is called the Hodge to de Rham spectral sequence for the Zariski-de Rham complex $\widetilde{\Omega}^\bullet_X(\log D)$.

Theorem 3.2. *Notation and assumptions as in Theorem 3.1, then the Hodge to de Rham spectral sequence (7) degenerates in E_1.*

Proof. By Theorem 2.4 and Proposition 2.5, there is a split injection of complexes:

$$\phi : \bigoplus_i \Omega^i_{U'}(\log D'|_{U'})[-i] \to F_*\Omega^\bullet_U(\log D|_U).$$

Applying ι'_* to ϕ, since $\iota'_*F_* = F_*\iota_*$, we have a split injection of complexes:

$$\iota'_*\phi : \bigoplus_i \widetilde{\Omega}^i_{X'}(\log D')[-i] \to F_*\widetilde{\Omega}^\bullet_X(\log D).$$

Thus we have

$$\begin{aligned} \sum_{i+j=n} \dim_k E_\infty^{ij} &= \dim_k \mathbf{H}^n(X, \widetilde{\Omega}^\bullet_X(\log D)) = \dim_k \mathbf{H}^n(X', F_*\widetilde{\Omega}^\bullet_X(\log D)) \\ &\geq \dim_k \mathbf{H}^n(X', \bigoplus_i \widetilde{\Omega}^i_{X'}(\log D')[-i]) = \sum_{i+j=n} \dim_k H^j(X', \widetilde{\Omega}^i_{X'}(\log D')) \\ &= \sum_{i+j=n} \dim_k H^j(X, \widetilde{\Omega}^i_X(\log D)) = \sum_{i+j=n} \dim_k E_1^{ij}. \end{aligned}$$

Since E_∞^{ij} is a subquotient of E_1^{ij}, we have $\sum_{i+j=n} \dim_k E_\infty^{ij} \leq \sum_{i+j=n} \dim_k E_1^{ij}$, which implies $E_\infty^{ij} \cong E_1^{ij}$ for any i, j, i.e. the Hodge to de Rham spectral sequence degenerates in E_1. $\square$

Lemma 3.3. *Notation and assumptions as in Theorem 3.1, let H be an ample $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}(\langle H \rangle) \subseteq D$. *Then there is an injection for any $r > 0$ and any $i, j \geq 0$:*

$$H^j(X, \widetilde{\Omega}^i_X(\log D)(-\ulcorner H \urcorner)) \hookrightarrow H^j(X, \widetilde{\Omega}^i_X(\log D)(-\ulcorner p^r H \urcorner)).$$

Proof. By Theorem 2.8, we have a split injection for any $i \geq 0$:

$$\phi^i_H : \Omega^i_{U'}(\log D'|_{U'})(-\ulcorner H'|_{U'} \urcorner) \to F_*(\Omega^i_U(\log D|_U)(-\ulcorner pH|_U \urcorner)).$$

Applying ι'_* to ϕ^i_H, since $\iota'_* F_* = F_* \iota_*$, we have a split injection for any $i \geq 0$:

$$\iota'_* \phi^i_H : \widetilde{\Omega}^i_{X'}(\log D')(-\ulcorner H' \urcorner) \to F_*(\widetilde{\Omega}^i_X(\log D)(-\ulcorner pH \urcorner)),$$

which gives rise to an injection for any $i, j \geq 0$:

$$H^j(X, \widetilde{\Omega}^i_X(\log D)(-\ulcorner H \urcorner)) \hookrightarrow H^j(X, \widetilde{\Omega}^i_X(\log D)(-\ulcorner pH \urcorner)).$$

Iterating these injections, we can obtain an injection for any $r > 0$ and any $i, j \geq 0$:

$$H^j(X, \widetilde{\Omega}^i_X(\log D)(-\ulcorner H \urcorner)) \hookrightarrow H^j(X, \widetilde{\Omega}^i_X(\log D)(-\ulcorner p^r H \urcorner)).$$

□

Corollary 3.4. *Let X be a smooth projective variety over k, D a simple normal crossing divisor on X, and H an ample $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}(\langle H \rangle) \subseteq D$. *Assume that (X, D) has a lifting $(\widetilde{X}, \widetilde{D})$ over $W_2(k)$ and $F : X \to X'$ has a lifting $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$, which is compatible with $\widetilde{D}$. Then*

$$H^j(X, \Omega^i_X(\log D)(-\ulcorner H \urcorner)) = 0$$

holds for any $j < \dim X$ and any $i \geq 0$.

Proof. By Lemma 3.3, there is an injection for any $r > 0$, any $i \geq 0$ and any $j < \dim X$:

$$H^j(X, \Omega^i_X(\log D)(-\ulcorner H \urcorner)) \hookrightarrow H^j(X, \Omega^i_X(\log D)(-\ulcorner p^r H \urcorner)). \tag{8}$$

Assume that mH is a Cartier divisor for some positive integer m. Assume $p^r = sm + t$, where $0 \leq t < m$. Then $\ulcorner p^r H \urcorner = s(mH) + \ulcorner tH \urcorner$. We can take r sufficiently large, such that s is sufficiently large. Since $\Omega^i_X(\log D)(-\ulcorner tH \urcorner)$ is locally free,

$$H^j(X, \Omega^i_X(\log D)(-\ulcorner p^r H \urcorner)) = H^j(X, \Omega^i_X(\log D)(-\ulcorner tH \urcorner) \otimes \mathcal{O}_X(mH)^{-s}) = 0 \tag{9}$$

holds for any $0 \leq t < m$ and any $j < \dim X$ by Serre's duality and Serre's vanishing. Thus the injection (8) implies the desired vanishing. □

It follows from [KM98, Theorem 5.71 and Corollary 5.72] that Serre's duality and the vanishing (9) hold only for Cohen-Macaulay sheaves on normal projective varieties. In general, the reflexive sheaves $\widetilde{\Omega}^i_X(\log D)(-\ulcorner tH \urcorner)$ are not necessarily Cohen-Macaulay, hence we cannot use the same argument to get a generalization of Corollary 3.4 to normal projective varieties. In order to deal with the Kawamata-Viehweg vanishing, it is helpful to introduce the following notion.

Definition 3.5. Let X be a normal projective variety of dimension n over k. We say that Serre's duality holds for any Weil divisor on X, if there is an isomorphism of k-vector spaces for any Weil divisor D on X and any $i \geq 0$:

$$H^i(X, \mathcal{O}_X(D))^\vee \xrightarrow{\sim} H^{n-i}(X, \mathcal{O}_X(K_X - D)),$$

where K_X is the canonical divisor of X (see [KM98, Proposition 5.75]).

Theorem 3.6. *Let X be a normal projective variety over k, D a reduced Weil divisor on X, and U a required open subset for (X, D). Assume that $(U, D|_U)$ has a lifting $(\widetilde{U}, \widetilde{D|_U})$ over $W_2(k)$ and $F : U \to U'$ has a lifting $\widetilde{F} : \widetilde{U} \to \widetilde{U}'$, which is compatible with $\widetilde{D|_U}$. Let H be an ample $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}(\langle H \rangle) \subseteq D$*. Assume that Serre's duality holds for any Weil divisor on X. Then $H^j(X, K_X + \ulcorner H \urcorner) = 0$ holds for any $j > 0$.*

Proof. Taking $i = 0$ in Lemma 3.3, we have an injection for any $r > 0$ and any $j < \dim X$:

$$H^j(X, \mathcal{O}_X(-\ulcorner H \urcorner)) \hookrightarrow H^j(X, \mathcal{O}_X(-\ulcorner p^r H \urcorner)). \tag{10}$$

Assume that mH is a Cartier divisor for some positive integer m. Assume $p^r = sm + t$, where $0 \leq t < m$. Then $\ulcorner p^r H \urcorner = s(mH) + \ulcorner tH \urcorner$. We can take r sufficiently large, such that s is sufficiently large. Let $n = \dim X$. By assumption, Serre's duality holds for any Weil divisor on X, hence we have

$$H^j(X, \mathcal{O}_X(-\ulcorner p^r H \urcorner)) \cong H^{n-j}(X, \mathcal{O}_X(K_X + \ulcorner tH \urcorner) \otimes \mathcal{O}_X(mH)^s)^\vee = 0$$

for any $0 \leq t < m$ and any $j < n$ by Serre's vanishing. The injection (10) implies that $H^j(X, \mathcal{O}_X(-\ulcorner H \urcorner)) = 0$ holds for any $j < n$, which implies that $H^j(X, K_X + \ulcorner H \urcorner) = 0$ holds for any $j > 0$ by Serre's duality again. ◻

Theorem 3.7. *Let $X = X(\Delta)$ be a toric variety over k. Then there exists a lifting $\widetilde{X}$ of X over $W_2(k)$ and a lifting $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ of the relative Frobenius morphism $F : X \to X'$ such that the following two conditions hold:*

(i) For any effective Cartier divisor E on X, there is a lifting $\widetilde{E} \subset \widetilde{X}$ of $E \subset X$, i.e. X is strongly liftable;

(ii) $\widetilde{F}$ is compatible with $\widetilde{E}$ which is taken in (i).

Proof. First of all, we recall some definitions and notation for toric varieties from [Fu93]. Let N be a lattice of rank n, and M the dual lattice of N. Let Δ be a fan consisting of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$, and let A be a ring. In general, we denote the toric variety associated to the fan Δ over the ground ring A by $X(\Delta, A)$. More precisely, to each cone σ in Δ, there is an associated affine toric variety $U_{(\sigma,A)} = \mathrm{Spec}\, A[\sigma^\vee \cap M]$, and these $U_{(\sigma,A)}$ can be glued together to form the toric variety $X(\Delta, A)$ over $\mathrm{Spec}\, A$. Note that almost all definitions, constructions and results for toric varieties are independent of the ground ring A, although everything is stated in [Fu93] over the complex number field $\mathbb{C}$.

In our case, $X = X(\Delta, k)$ is the toric variety over k associated to the fan Δ. Let $\widetilde{X} = X(\Delta, W_2(k))$. Note that $U_{(\sigma,k)} = \mathrm{Spec}\, k[\sigma^\vee \cap M]$, $U_{(\sigma,W_2(k))} = \mathrm{Spec}\, W_2(k)[\sigma^\vee \cap M]$ is flat over $W_2(k)$ and $U_{(\sigma,W_2(k))} \times_{\mathrm{Spec}\, W_2(k)} \mathrm{Spec}\, k = U_{(\sigma,k)}$, hence $\widetilde{X}$ is a lifting of X over $W_2(k)$.

By definition, to lift the relative Frobenius morphism $F : X \to X'$ over $W_2(k)$, we have only to lift the absolute Frobenius morphism $F_X : X \to X$ over $W_2(k)$. On each affine piece $U_{(\sigma,W_2(k))}$, define $\widetilde{F}_\sigma : U_{(\sigma,W_2(k))} \to U_{(\sigma,W_2(k))}$ by $F_{W_2(k)} : W_2(k) \to W_2(k)$ and $\sigma^\vee \cap M \to \sigma^\vee \cap M$, $u \mapsto pu$. It is easy to see that we can glue $\widetilde{F}_\sigma$ together to obtain a morphism $\widetilde{F}_X : \widetilde{X} \to \widetilde{X}$ lifting $F_X : X \to X$.

Let $\mathcal{L} = \mathcal{O}_X(E)$ be the associated invertible sheaf on X. By [Fu93, Page 63, Proposition], we have an exact sequence:

$$0 \to M \to \mathrm{Div}_T(X) \to \mathrm{Pic}(X) \to 0.$$

Thus there exists a torus invariant Cartier divisor D on X such that E is linearly equivalent to D. Assume that $\{u(\sigma) \in M/M(\sigma)\} \in \varprojlim M/M(\sigma)$ determines the torus invariant Cartier divisor D. Then the same data $\{u(\sigma) \in M/M(\sigma)\}$ also determines a torus invariant Cartier divisor $\widetilde{D}$ on $\widetilde{X}$ (we have only to change the base k into $W_2(k)$). Thus the invertible sheaf $\mathcal{L} = \mathcal{O}_X(E) = \mathcal{O}_X(D)$ has a lifting $\widetilde{\mathcal{L}} = \mathcal{O}_{\widetilde{X}}(\widetilde{D})$ on $\widetilde{X}$. Let v_i be the first lattice points in the edges of the maximal dimensional cones in Δ, D_i the corresponding orbit closures in X, and $\widetilde{D}_i$ the corresponding orbit closures in $\widetilde{X}$ $(1 \leq i \leq N)$. Then the torus invariant Cartier divisors $D = \sum_{i=1}^N a_i D_i$ and $\widetilde{D} = \sum_{i=1}^N a_i \widetilde{D}_i$ determine a rational convex polyhedral P_D in $M_\mathbb{R}$ defined by

$$P_D = \{u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geq -a_i,\ 1 \leq i \leq N\}.$$

By [Fu93, Page 66, Lemma], we have

$$H^0(X, D) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u,\ H^0(\widetilde{X}, \widetilde{D}) = \bigoplus_{u \in P_D \cap M} W_2(k) \cdot \chi^u.$$

Thus the map $H^0(\widetilde{X}, \widetilde{D}) \xrightarrow{r} H^0(X, D)$ induced by the natural surjection $W_2(k) \xrightarrow{r} k$ is obviously surjective. Hence $r : H^0(\widetilde{X}, \widetilde{\mathcal{L}}) \to H^0(X, \mathcal{L})$ is surjective. By Proposition 2.10, X is strongly liftable over $W_2(k)$.

By construction, assume that $\widetilde{E} \subset \widetilde{X}$ is a lifting of $E \subset X$ such that $\widetilde{E}$ is linearly equivalent to $\widetilde{D}$. By the definition of the lifting $\widetilde{F}_X : \widetilde{X} \to \widetilde{X}$, we have $\widetilde{F}_X^* \mathcal{O}_{\widetilde{X}}(-\widetilde{D}) = \mathcal{O}_{\widetilde{X}}(-p\widetilde{D})$ holds, hence $\widetilde{F}_X^* \mathcal{O}_{\widetilde{X}}(-\widetilde{E}) = \mathcal{O}_{\widetilde{X}}(-p\widetilde{E})$ holds. □

Corollary 3.8. *Let $X = X(\Delta, k)$ be a toric variety, and D a reduced Weil divisor on X. Then there is a required open subset U for (X, D), such that $(U, D|_U)$ has a lifting $(\widetilde{U}, \widetilde{D|_U})$ over $W_2(k)$ and $F : U \to U'$ has a lifting $\widetilde{F} : \widetilde{U} \to \widetilde{U}'$, which is compatible with $\widetilde{D|_U}$.*

Proof. Let $V = X(\Delta_1)$, where Δ_1 is the fan consisting of all 1-dimensional cones in Δ. Then it is easy to see that V is an open subset of X with $\mathrm{codim}_X(X - V) \geq 2$, V is a smooth toric variety over k, and $D|_V$ is a reduced Cartier divisor on V.

By Theorem 3.7, $(V, D|_V)$ has a lifting $(\widetilde{V}, \widetilde{D|_V})$ over $W_2(k)$ and $F : V \to V'$ has a lifting $\widetilde{F} : \widetilde{V} \to \widetilde{V}'$, which is compatible with $\widetilde{D|_V}$. Shrink V into U if necessary, such that U is an open subset of X with $\mathrm{codim}_X(X - U) \geq 2$, U is smooth over k and $D|_U$ is simple normal crossing on U. Then $(U, D|_U)$ has a lifting $(\widetilde{U}, \widetilde{D|_U})$ over $W_2(k)$ and $F : U \to U'$ has a lifting $\widetilde{F} : \widetilde{U} \to \widetilde{U}'$, which is compatible with $\widetilde{D|_U}$. □

Now, the main theorems are easy consequences of the above more general results.

Proof of Theorem 1.1. It follows from Theorem 3.1 and Corollary 3.8. □

Proof of Theorem 1.2. It follows from Theorem 3.2 and Corollary 3.8. □

Proof of Theorem 1.3. It follows from Theorem 3.6, Corollary 3.8 and toric Serre's duality [CLH11, Theorem 9.2.10(a) and Remark 9.2.11(b)]. ☐

The following corollary shows that the main theorems could hold for more general varieties which are not necessarily toric.

Corollary 3.9. *Let X be a smooth projective toric variety with* $\dim X \geq 2$*, and $g : Y \to X$ the composition of blow-ups along closed points. Then the main theorems hold for Y.*

Proof. Let $f : X_1 \to X$ be the blow-up of X along a closed point P. If we can prove the liftability of the relative Frobenius morphism and the strong liftability of X_1, then the main theorems hold for X_1, hence hold for Y by induction.

By Theorem 3.7, we have a lifting $\widetilde{X}$ of X over $W_2(k)$ and a lifting $\widetilde{F}_X : \widetilde{X} \to \widetilde{X}$ of $F_X : X \to X$ compatible with any lifting of divisors on $\widetilde{X}$. By [Xie10, Lemma 2.5], $P \in X$ has a lifting $\widetilde{P} \in \widetilde{X}$. Let $\widetilde{f} : \widetilde{X}_1 \to \widetilde{X}$ be the blow-up of $\widetilde{X}$ along $\widetilde{P}$, $\widetilde{E}$ the exceptional divisor of $\widetilde{f}$, and $\mathcal{I}_{\widetilde{P}}$ the ideal sheaf of $\widetilde{P} \in \widetilde{X}$. Consider the following diagram:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[r, dotted, "\widetilde{F}_{X_1}"] \arrow[d, "\widetilde{f}"'] & \widetilde{X}_1 \arrow[d, "\widetilde{f}"] \\ \widetilde{X} \arrow[r, "\widetilde{F}_X"] & \widetilde{X}. \end{tikzcd}$$

Since $(\widetilde{f}\widetilde{F}_X)^{-1}\mathcal{I}_{\widetilde{P}} \cdot \mathcal{O}_{\widetilde{X}_1} = \mathcal{O}_{\widetilde{X}_1}(-p\widetilde{E})$ is an invertible sheaf of ideals on $\widetilde{X}_1$, by the universal property of blow-up, there is a unique morphism $\widetilde{F}_{X_1} : \widetilde{X}_1 \to \widetilde{X}_1$ making the above diagram commutative. It is easy to verify that $\widetilde{F}_{X_1} : \widetilde{X}_1 \to \widetilde{X}_1$ is a lifting of $F_{X_1} : X_1 \to X_1$ compatible with any lifting of divisors on $\widetilde{X}_1$.

On the other hand, by [Xie10, Proposition 2.6], X_1 is strongly liftable over $W_2(k)$, which completes the proof. ☐

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School of Mathematical Sciences, Fudan University, Shanghai 200433, China
E-mail address: qhxie@fudan.edu.cn